Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Fuzzy soft upper and lower semi-continuous multifunctions

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Received 22 May 2018; Revised 10 June 2018; Accepted 13 August 2018

Abstract. In this paper, we define the upper and lower inverse of a fuzzy soft multifunction and prove some basic identities. Then by using these ideas we introduce the concept of fuzzy soft continuity and obtain many interesting properties of fuzzy soft upper and lower semi-continuous multifunctions. Also, we define the notions of fuzzy soft lower $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuity and fuzzy soft upper $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuity. Moreover, we present different cases of upper and lower fuzzy soft continuous multifunctions.

2010 AMS Classification: 54A40, 54C10, 54D05, 54D10, 54D30

Keywords: Fuzzy soft set, Fuzzy soft multifunctions, Fuzzy soft continuity.

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1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are characteristically crisp, deterministic and precise. However, in real life, the problems in economics, engineering, environment, social science, medical science, etc., do not always involve crisp data. The reason for these difficulties is possibly, the inadequacy of the classical parameterization tool in general. Consequently, Molodtsov [17] initiated the concept of soft set theory as a new mathematical tool for dealing with vagueness and uncertainties which is free from the above difficulties. Molodtsov [18] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration and theory of measurement. Maji et al.[14] gave a practical application of soft sets in decisionmaking problems. They have also introduced the concept of fuzzy soft set (Maji et al. [13]), as more generalized concept, which is a combination of fuzzy set (Zadeh [28]) and soft set (Molodtsov [17]) and also studied some of its properties. Many of applications can be found in the works of Ahmad and Kharal [7], Kharal and Ahmad [12], Tanay and Kandemir [27], Aygunoglu et al. [8], Cetkin et al. [9], Metin et al. [15], Metin and Alkan [16], Abbas et al. [1, 2, 3], Gunduz and Bayramov [11], Abbas and Ibedou [6], Dizmana et al. [10], Qiu and Zhang [19], Qiu et al. [20, 21] and Šenel [22, 23, 24], Šenel and Cagman [25] and Serkan and Idris [26]. Metin et al. ([15]) defined the upper and lower inverse of a fuzzy soft multi-function from ordinary topological space to fuzzy soft topological space in Chang sense and proved some basic properties. Although there are some similarities of the definitions and theorems between fuzzy soft topological spaces and fuzzy topological spaces, there are a lot of differences. They arise especially when the cardinality $|E|$ of the set E of parameters is greater than 1. If $|E| = 1$, then a fuzzy soft topological spaces can be treated as a fuzzy topological spaces and then the behavior of fuzzy soft topological space τ_E is analogous to a fuzzy topological space τ.

In this paper our purpose is two fold. First, we introduce the notions of fuzzy soft multifunction, upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we introduce fuzzy soft upper and lower semi-continuous multifunctions. Also, we define the notions of fuzzy soft lower $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuous and fuzzy soft upper $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuous, then we present different cases of the fuzzy soft continuity multifunction.

2. Preliminaries

Throughout this paper, X refers to an initial universe and E is the set of all parameters for X. A fuzzy soft (simply, FS) set f_E on X is called λ-absolute FS set and denoted by $\tilde{\mathrm{E}}^{\lambda}$, if $f_e = \underline{\lambda}$, for $\lambda \in I$, $\underline{\lambda}(x) = \lambda$, for each $x \in X$ and $e \in E$ (where $I = [0,1], I_0 = (0,1]$). For $f_A, g_B \in \widetilde{(X,E)}$, $f_A \,\bar{\wedge}\, g_B = \Phi$ if $f_A \sqsubseteq g_B$ and $f_A \,\bar{\wedge}\, g_B = f_A \sqcap g^c_B$ otherwise. The FS set $f_A = e^t_x = \{(e, x_t)\}$ is called FS point where $t \in I_0$, x_t is fuzzy point and we say that e^t_x belongs to the FS set f_A if for $e \in A$, $t \leq f_e(x)$. Also, the map $\alpha : E \times \widetilde{(X,E)} \times I_0 \to \widetilde{(X,E)}$ is called a FS operator on X.

All definitions and properties of FS sets and FS topology are found in [1, 2, 8, 9, 10, 12, 14, 15, 21]. Recall that a FS ideal $\mathcal{I}$ on X is a mapping $\mathcal{I} : E \to I^{\widetilde{(X,E)}}$ that satisfies the following conditions for each $e \in E$,

(i) $\mathcal{I}_e(\Phi) = 1,\ \mathcal{I}_e(\tilde{\mathrm{E}}) = 0$,
(ii) $\mathcal{I}_e(f_A \sqcup g_B) \geq \mathcal{I}_e(f_A) \wedge \mathcal{I}_e(g_B$, for each $f_A, g_B \in \widetilde{(X,E)}$,
(iii) if $f_A \sqsubseteq g_B$, then $\mathcal{I}_e(f_A) \geq \mathcal{I}_e(g_B)$.

For $f_A \in \widetilde{(X,E)}$, $\mathcal{I}^0_e(f_A) = 1$, if $f_A = \Phi$ and $\mathcal{I}^0_e(f_A) = 0$, otherwise.

(X, τ_E) is a FS topological space (See [9]), if a mapping $\tau : E \to I^{\widetilde{(X,E)}}$ satisfies the following conditions for each $e \in E$,

(T1) $\tau_e(\Phi) = \tau_e(\tilde{\mathrm{E}}) = 1$,
(T2) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$, $\forall\ f_A, g_B \in \widetilde{(X,E)}$,
(T3) $\tau_e(\bigsqcup_{i\in\Gamma}(f_A)_i) \geq \bigwedge_{i\in\Gamma} \tau_e((f_A)_i)$, $\forall\ (f_A)_i \in \widetilde{(X,E)}$, $i \in \Gamma$.

Let (X, τ_E) be a FS topological space. Then the mapping $\tau^e : I^X \to I$ defined as: for each $e \in E$,

$$\tau^e(\lambda) = \begin{cases} & \vee\{r \in I_0 : \tau_e(f_A) \geq r\}, \ \lambda = f_e \\ 0 & \text{otherwise} \end{cases}$$

is a fuzzy topology on X, so for each parameter $e \in E$, we have a fuzzy topology τ^e on X. Thus, a FS topology on X gives a parameterized famely of fuzzy topologies on X.

3. Fuzzy soft multifunctions.

In this section we introduce the same definition from FS topological space to FS topological space in Šostak sense which is a generalization of the same concepts introduced by Metin et al. 2015 ([17]).

Definition 3.1. Let $\psi : E \to K$. Then, $F : X \multimap Y$ is called a FS multifunction, if $F(x) : K \to I^Y$ is a FS set on $\widetilde{(Y, K)}$, for each $x \in X$. Also, F is said to be onto, if for each FS set $g_B \in \widetilde{(Y, K)}$, there is $x \in X$ such that $F(x) = g_B$.

The degree of membership of y in $F(x)$ with respect to a parameter $k \in K$ is denoted by:

$$(F(x))_k(y) = (G_F)_k(x, y), \quad k \in K,$$

where $G_F : K \to I^{X \times Y}$.

The inverse of F denoted by $F^- : Y \multimap X$ is a FS multifunction defined by:

$$(F^-(y))_e(x) = (F(x))_{\psi(e)}(y) = (G_F)_{\psi(e)}(x, y), \ e \in E.$$

The domain of F denoted by $dom(F) : E \to I^X$, is defined as:

$$(dom(F))_e(x) = \bigvee_{y \in Y} (G_F)_{\psi(e)}(x, y), \quad \text{for } x \in X, \ y \in Y \ \text{ and } \ e \in E.$$

The range of F denoted by $rng(F) : K \to I^Y$, is defined as:

$$(rng(F))_k(y) = \bigvee_{x \in X} (G_F)_k(x, y), \quad \text{for } x \in X, \ y \in Y \ \text{ and } k \in K.$$

Definition 3.2. FS multifunction $F : X \multimap Y$ is called:

(i) non-void, if $F(x) \neq \Phi, \ \forall x \in X$,
(ii) surjective, if $(rng(F))_k(y) = 1, \ \forall y \in Y, \ k \in K, \ \psi : E \to K$.
(iii) normalized, if $\forall x \in X, \ e \in E$ there is $y_0 \in Y$ such that $(G_F)_e(x, y_0) = 1$.

Definition 3.3. Let $F : X \multimap Y$ be a FS multifunction and $\psi : E \to K$. Then

(i) the image $F(f_A)$ of $f_A \in \widetilde{(X, E)}$ is FS set in $\widetilde{(Y, K)}$ defined as:

$$(F(f_A))_k(y) = \bigvee_{x \in X} [(G_F)_k(x, y) \wedge f_{\psi^{-1}(k)}(x)],$$

(ii) the lower inverse $F^l(g_B)$ of $g_B \in \widetilde{(Y,K)}$ is FS set in $\widetilde{(X,E)}$ defined as:

$$(F^l(g_B))_e(x) = \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y) \wedge g_{\psi(e)}(y)],$$

(iii) the upper inverse $F^u(g_B)$ of $g_B \in \widetilde{(Y,K)}$ is FS set in $\widetilde{(X,E)}$ defined as:

$$F^u(g_B)(e)(x) = \bigwedge_{y\in Y}[(G^c_F)_{\psi(e)}(x,y) \vee g_{\psi(e)}(y)].$$

Example 3.4. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $\psi : E = \{e_1, e_2\} \to \{k_1, k_2\} = K$, where $\psi(e_1) = k_1$, $\psi(e_2) = k_2$. And let $F : X \multimap Y$ be a FS multifunction defined as:

$(G_F)_{k_1}(x_1, y_1) = 0.2$, $(G_F)_{k_1}(x_1, y_2) = 0.3$, $(G_F)_{k_1}(x_2, y_1) = 0.4$,
$(G_F)_{k_1}(x_2, y_2) = 0.2$, $(G_F)_{k_2}(x_1, y_1) = 0.5$, $(G_F)_{k_2}(x_1, y_2) = 1.0$,
$(G_F)_{k_2}(x_2, y_1) = 0.0$, $(G_F)_{k_2}(x_2, y_2) = 0.7$.

Then

$$dom(F) = \{(e_1, \{0.3, 0.4\}), (e_2, \{1.0, 0.7\})\}$$

and

$$rng(F) = \{(k_1, \{0.4, 0.3\}), (k_2, \{0.5, 1.0\})\}.$$

Thus for FS sets,

$$f_E = \{(e_1, \{0.2, 0.8\}), (e_2, \{0.3, 0.5\})\}$$

and

$$g_K = \{(k_1, \{0.7, 0.1\}), (k_2, \{0.6, 0.5\})\},$$

we have

$$F(f_E) = \{(k_1, \{0.4, 0.2\}), (k_2, \{0.3, 0.5\})\},$$
$$F^l(g_K) = \{(k_1, \{0.2, 0.4\}), (k_2, \{0.5, 0.5\})\}$$

and

$$F^u(g_K) = \{(k_1, \underline{0.7}), (k_2, \underline{0.5})\}.$$

So $F^u(g_B) \neq F^l(g_B)$.

Proposition 3.5. *Let $F : X \multimap Y$ be a FS multifunction and $\psi : E \to K$. Then*

(1) $F((f_A)_1) \sqsubseteq F((f_A)_2)$ *if* $(f_A)_1 \sqsubseteq (f_A)_2$,
(2) $F^l((g_B)_1) \sqsubseteq F^l((g_B)_2)$ *and* $F^u((g_B)_1) \sqsubseteq F^u((g_B)_2)$, *if* $(g_B)_1 \sqsubseteq (g_B)_2$,
(3) $F^u(g_B) \sqsubseteq F^l(g_B)$ *and* $(F^l(g_B))^c \sqsubseteq F^l(g^c_B)$, *if F is normalized,*
(4) $(F(f_A))^c \sqsubseteq F(f^c_A)$, *if F is surjective,*
(5) $(F^l(g^c_B))^c = F^u(g_B)$ *and* $(F^u(g^c_B))^c = F^l(g_B)$,
(6) $F(\tilde{E}^{\lambda}) = \tilde{K}^{\lambda}$, *if F is surjective,*
(7) $F^l(g_B \sqcap \tilde{K}^{\lambda}) = F^l(g_B) \sqcap \tilde{E}^{\lambda}$.

Proof. (1) and (2) obvious from the definition.

(3) Since F is normalized, there exists $y_0 \in Y$ such that $(G_F)_{\psi(e)}(x, y_0) = 1$. Then

$$\begin{aligned}(F^l(g_B))_e(x) &= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y) \wedge g_{\psi(e)}(y)] \\ &\geq [(G_F)_{\psi(e)}(x,y_0) \wedge g_{\psi(e)}(y_0)] = g_{\psi(e)}(y_0),\end{aligned}$$

$$\begin{aligned}(F^u(g_B))_e(x) &= \bigwedge_{y\in Y}[(G^c_F)_{\psi(e)}(x,y)\vee g_{\psi(e)}(y)]\\ &\leq [(G^c_F)_{\psi(e)}(x,y_0)\vee g_{\psi(e)}(y_0)] = g_{\psi(e)}(y_0).\end{aligned}$$

Thus $F^u(g_B)\sqsubseteq F^l(g_B)$ and

$$\begin{aligned}((F^l(g_B))_e(x))^c &= (\bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge g_{\psi(e)}(y)])^c\\ &= \bigwedge_{y\in Y}[(G^c_F)_{\psi(e)}(x,y)\vee g^c_{\psi(e)}(y)]\\ &\leq [(G^c_F)_{\psi(e)}(x,y_0)\vee g^c_{\psi(e)}(y_0)] = g^c_{\psi(e)}(y_0),\end{aligned}$$

$$\begin{aligned}(F^l(g^c_B))_e(x) &= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge g^c_{\psi(e)}(y)]\\ &\geq [(G_F)_{\psi(e)}(x,y_0)\wedge g^c_{\psi(e)}(y_0)] = g^c_{\psi(e)}(y_0).\end{aligned}$$

So $(F^l(g_B))^c\sqsubseteq F^l(g^c_B)$.

(4) By similar way as (3).

(5) $((F^l(g^c_B))_e(x))^c = (\bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge g^c_{\psi(e)}(y)])^c$
$= \bigwedge_{y\in Y}[(G^c_F)_{\psi(e)}(x,y)\vee g_{\psi(e)}(y)] = (F^u(g_B))_e)(x)$.

Then $(F^l(g^c_B))^c = F^u(g_B)$.

By similar way, we can prove that $(F^u(g^c_B))^c = F^l(g_B)$.

(6) Since $(F(\tilde{\mathrm{E}}^\lambda))_k(y) = \bigvee_{x\in X}[(G_F)_k(x,y)\wedge \tilde{\mathrm{E}}^\lambda_{\psi^{-1}(k)}(x)]$
$= \bigwedge_{y\in Y}[(G^c_F)_{\psi(e)}(x,y)\vee g_{\psi(e)}(y)]$
$= (F^u(g_B))_e)(x)$,

$F(\tilde{\mathrm{E}}^\lambda) = \tilde{\mathrm{K}}^\lambda$.

(7) Since $(F^l(g_B\sqcap\tilde{\mathrm{K}}^\lambda))_e(x) = \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge(g_B\sqcap\tilde{\mathrm{K}}^\lambda)_{\psi(e)}(y)]$
$= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge(g_{\psi(e)}(y)]\wedge\lambda$
$= [(F^l(g_B))_e(x)]\wedge\lambda$,

$F^l(g_B\sqcap\tilde{\mathrm{K}}^\lambda) = F^l(g_B)\sqcap\tilde{\mathrm{E}}^\lambda$. □

Proposition 3.6. *Let* $F: X\multimap Y$ *be a FS multifunction and* $\psi: E\to K$. *Then*

(1) $F(\sqcap_{i\in\Gamma}(f_A)_i)\sqsubseteq\sqcap_{i\in\Gamma}F((f_A)_i)$ *and* $F(\sqcup_{i\in\Gamma}(f_A)_i) = \sqcup_{i\in\Gamma}F((f_A)_i)$,
(2) $F^l(\sqcap_{i\in\Gamma}(g_B)_i)\sqsubseteq\sqcap_{i\in\Gamma}F^l((g_B)_i)$ *and* $F^l(\sqcup_{i\in\Gamma}(g_B)_i) = \sqcup_{i\in\Gamma}F^l((g_B)_i)$,
(3) $F^u(\sqcup_{i\in\Gamma}(g_B)_i)\sqsupseteq\sqcup_{i\in\Gamma}F^u((g_B)_i)$ *and* $F^u(\sqcap_{i\in\Gamma}(g_B)_i) = \sqcap_{i\in\Gamma}F^u((g_B)_i)$.

Proof.

$$
\begin{aligned}
(1)\quad (F(\sqcap_{i\in\Gamma}(f_A)_i))_k(y) &= \bigvee_{x\in X}[(G_F)_k(x,y)\wedge(\sqcap_{i\in\Gamma}(f_A)_i)_{\psi^{-1}(k)}(x)]\\
&= \bigvee_{x\in X}[\bigwedge_{i\in\Gamma}(G_F)_k(x,y)\wedge((f_A)_i)_{\psi^{-1}(k)}(x)]\\
&\leq \bigwedge_{i\in\Gamma}[\bigvee_{x\in X}(G_F)_k(x,y)\wedge((f_A)_i)_{\psi^{-1}(k)}(x)]\\
&= \bigwedge_{i\in\Gamma}(F((f_A)_i))_k(y).
\end{aligned}
$$

Then $F(\sqcap_{i\in\Gamma}(f_A)_i)\sqsubseteq\sqcap_{i\in\Gamma}F((f_A)_i)$.

By the same way, we can prove (2) and (3). □

Proposition 3.7. *Let $F: X \multimap Y$ be a FS multifunction and $\psi: E\to K$. Then*

(1) $F(F^l(g_B))\sqsupseteq g_B$ *and* $F(F^u(g_B))\sqsubseteq g_B$, *if* F *is surjective,*
(2) $F^l(F(f_A))\sqsupseteq f_A$ *and* $F^u(F(f_A))\sqsupseteq f_A$, *if* F *is normalized,*
(3) $F(F^u(g_B))\neq g_B$, $F^u(F(f_A))\neq f_A$ *and* $F(F^l(g_B))\neq g_B$.

Proof. (1) Since F is surjective, we have $rng(F)_k(y)=1$, for all $y\in Y$, $k\in K$. Then

$$
\begin{aligned}
(F(F^l(g_B)))_k(y) &= \bigvee_{x\in X}[(G_F)_k(x,y)\wedge(F^l(g_B))_{\psi^{-1}(k)}(x)]\\
&= \bigvee_{x\in X}[(G_F)_k(x,y)\wedge(\bigvee_{y\in Y}[(G_F)_k(x,y)\wedge g_k(y))]]\\
&\geq \bigvee_{x\in X}[(G_F)_k(x,y)\wedge((G_F)_k(x,y)\wedge g_k(y))]\\
&= \bigvee_{x\in X}[(G_F)_k(x,y)\wedge g_k(y)] = g_k(y).
\end{aligned}
$$

Thus $F(F^l(g_B))\sqsupseteq g_B$.

The other case is similarly.

(2) Since F is normalized, there exists $y_0\in Y$ such that $(G_F)_k(x,y_0)=1$. Then

$$
\begin{aligned}
(F^l(F(f_A)))_e(x) &= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge(F(f_A))_{\psi(e)}(y)]\\
&= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge(\bigvee_{x\in X}((G_F)_k(x,y)\wedge f_e(x)))]\\
&\geq \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge((G_F)_k(x,y_0)\wedge f_e(x))]\\
&= \bigvee_{y\in Y}[(G_F)_{\psi(e)}(x,y)\wedge f_e(x)] = f_e(x).
\end{aligned}
$$

Thus $F^l(F(f_A))\sqsupseteq f_A$.

The other case is similarly. □

Example 3.8. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $\psi : E = \{e_1, e_2\} \to \{k_1, k_2\} = K$, where $\psi(e_1) = k_1$, $\psi(e_2) = k_2$. And let $F : X \multimap Y$ be a FS multifunction defined by:

$(G_F)_{k_1}(x_1, y_1) = 0.1$, $(G_F)_{k_1}(x_1, y_2) = 1.0$, $(G_F)_{k_1}(x_2, y_1) = 0.0$,
$(G_F)_{k_1}(x_2, y_2) = 0.5$, $(G_F)_{k_2}(x_1, y_1) = 0.3$, $(G_F)_{k_2}(x_1, y_2) = 0.5$,
$(G_F)_{k_2}(x_2, y_1) = 0.2$, $(G_F)_{k_2}(x_2, y_2) = 0.0$.

If $f_E = \{(e_1, \{0.3, 0.4\}), (e_2, \underline{0.7})\}$, then

$$F^l(F(f_E)) = \{(e_1, \underline{0.4}), (e_2, \{0.5, 0.2\})\} \neq f_E$$

and

$$F^u(F(f_E)) = \{(e_1, \{0.4, 0.5\}), (e_2, \{0.5, 0.8\})\} \neq f_E.$$

If $g_K = \{(k_1, \{0.6, 0.4\}), (k_2, \{1.0, 0.3\})\}$, then

$$F(F^l(g_K)) = \{(k_1, \{0.1, 0.4\}), (k_2, \underline{0.3})\} \neq g_K$$

and

$$F(F^u(g_K)) = \{(k_1, \{0.1, 0.5\}), (k_2, \{0.3, 0.5\})\} \neq g_K.$$

Definition 3.9. Let $\psi : E \to K$, $\varpi : K \to M$ and FS multifunctions $F : X \multimap Y$, $H : Y \multimap Z$. Then the composition $H \circ F$ defined as:

$$((H \circ F)(x))_m(z) = \bigvee_{y \in Y} [(G_H)_m(y, z) \wedge (G_F)_{\varpi^{-1}(m)}(x, y)].$$

Theorem 3.10. *Let* $\psi : E \to K$, $\varpi : K \to M$ *and FS multifunctions* $F : X \multimap Y$, $H : Y \multimap Z$. *Then*

(1) $((H \circ F) = H(F)$,
(2) $(H \circ F)^l = F^l(H^l)$,
(3) $(H \circ F)^u = F^u(H^u)$.

Proof. (1) Let $f_A \in \widetilde{(X, E)}$. Then

$$\begin{aligned}
((H \circ F)(f_A))_m(z) &= \bigvee_{x \in X} [(G_{H \circ F})_m(x, z) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x)] \\
&= \bigvee_{x \in X} [\bigvee_{y \in Y} ((G_H)_m(y, z) \wedge (G_F)_{\varpi^{-1}(m)}(x, y)) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x)] \\
&= \bigvee_{y \in Y} [G_H)_m(y, z) \wedge (\bigvee_{x \in X} ((G_F)_{\varpi^{-1}(m)}(x, y) \wedge f_{\psi^{-1}(\varpi^{-1}(m))}(x)))] \\
&= \bigvee_{y \in Y} [(G_H)_m(y, z) \wedge (F(f_A))_{\varpi^{-1}(m)}(y)] \\
&= (H(F(f_A)))_m(z).
\end{aligned}$$

Thus $((H \circ F) = H(F)$.

(2) Let $w_D \in \widetilde{(Z,M)}$. Then

$$\begin{aligned}((H\circ F)^l(w_D))_e(x) &= \bigvee_{z\in Z}[(G_{H\circ F})_{\varpi(\psi(e))}(x,z)\wedge w_{\varpi(\psi(e))}(z)]\\ &= \bigvee_{z\in Z}[\bigvee_{y\in Y}((G_F)_{\psi(e)}(x,y)\wedge (G_H(y,z))_{(\varpi(\psi(e))})\wedge w_{\varpi(\psi(e))}(z)]\\ &= \bigvee_{y\in Y}[G_F)_{\psi(e)}(x,y)\wedge (\bigvee_{z\in Z}[(G_H)_{\varpi(\psi(e))}(y,z)\wedge w_{\varpi(\psi(e))}(z))]\\ &= \bigvee_{y\in Y}[G_F)_{\psi(e)}(x,y)\wedge (H^l(w_D))_{\psi(e)}(y)]\\ &= (F^l(H^l(w_D)))_e(x).\end{aligned}$$

Thus $(H\circ F)^l = F^l(H^l)$.

(3) The proof is similar to (2). □

4. Continuity of fuzzy soft multifunctions.

In this section, we introduce FS upper and lower semi-continuous multifunctions which a generalization of FS continuous multifunctions introduced by Metin et al. 2015 ([17]).

Definition 4.1. Let $\psi : E \to K$ and $F : X \multimap Y$ be a FS multifunction between two FS topological spaces (X,τ_E), (Y,η_K), $e\in E$ and $r\in I_0$. Then F is called:

- (i) FS upper semi-continuous at a FS point $e^t_x \in dom(F)$, if $e^t_x \in F^u(g_B)$ for each $g_B \in \widetilde{(Y,K)}$ and $\eta_{\psi(e)}(g_B)\geq r$, there exists $f_A \in \widetilde{(X,E)}$, $\tau_e(f_A)\geq r$ and $e^t_x \in f_A$ such that $f_A \sqcap dom(F) \sqsubseteq F^u(g_B)$,
- (ii) FS upper semi-continuous, if it is FS upper semi-continuous at every $e^t_x \in dom(F)$,
- (iii) FS lower semi-continuous at a FS point $e^t_x \in dom(F)$, if $e^t_x \in F^l(g_B)$ for each $g_B \in \widetilde{(Y,K)}$ and $\eta_{\psi(e)}(g_B)\geq r$, there exists $f_A \in \widetilde{(X,E)}$, $\tau_e(f_A)\geq r$ and $e^t_x \in f_A$ such that $f_A \sqsubseteq F^l(g_B)$,
- (iv) FS lower semi-continuous, if it is FS lower semi-continuous at every $e^t_x \in dom(F)$.

Remark 4.2. If F is normalized, then F is FS upper semi-continuous at a FS point $e^t_x \in dom(F)$ iff $e^t_x \in F^u(g_B)$ for each $g_B \in \widetilde{(Y,K)}$ and $\eta_{\psi(e)}(g_B)\geq r$, there exists $f_A \in \widetilde{(X,E)}$, $\tau_e(f_A)\geq r$ and $e^t_x \in f_A$ such that $f_A \sqsubseteq F^u(g_B)$.

Theorem 4.3. *Let $F : X \multimap Y$ be a FS multifunction between two FS topological spaces (X,τ_E), (Y,η_K), $\psi : E\to K$, $g_B \in \widetilde{(Y,K)}$ and $e\in E$.*

(1) *If F is normalized, then F is FS upper semi-continuous iff*

$$\tau_e(F^u(g_B)) \geq \eta_{\psi(e)}(g_B).$$

(2) *F is FS lower semi-continuous iff*

$$\tau_e(F^l(g_B)) \geq \eta_{\psi(e)}(g_B).$$

(3) *If* $F : (X, \tau_E) \multimap (Y, \eta_K)$ *is FS lower semi-continuous multifunction, then* $F : (X, \tau^e) \multimap (Y, \eta^{\psi(e)})$ *is fuzzy lower semi-continuous multifunction, for each* $e \in E$ *[4].*

(4) *If the normalized* $F : (X, \tau_E) \multimap (Y, \eta_K)$ *is FS upper semi-continuous multifunction, then* $F : (X, \tau^e) \multimap (Y, \eta^{\psi(e)})$ *is fuzzy upper semi-continuous multifunction, for each* $e \in E$.

Proof. (1) Assume that, there is $g_B \in \widetilde{(Y, K)}$ and $r \in I_0$ such that

$$\tau_e(F^u(g_B)) \leq r < \eta_{\psi(e)}(g_B).$$

Since F is normalized and FS upper semi-continuous, for $e_x^t \in F^u(g_B)$ with $\eta_{\psi(e)}(g_B) \geq r$, there is $(f_A)_{e_x^t} \in \widetilde{(X, E)}$ with $\tau_e((f_A)_{e_x^t}) \geq r$ and $e_x^t \in (f_A)_{e_x^t}$ such that $(f_A)_{e_x^t} \sqsubseteq F^u(g_B)$. Then $e_x^t \in F^u(g_B) = \bigsqcup_{e_x^t \in F^u(g_B)} (f_A)_{e_x^t}$. Thus

$$\tau_e(F^u(g_B)) = \tau_e(\bigsqcup_{e_x^t \in F^u(g_B)} (f_A)_{e_x^t}) \geq \bigwedge_{e_x^t \in F^u(g_B)} \tau_e((f_A)_{e_x^t}) \geq r.$$

It is a contradiction. So $\tau_e(F^u(g_B)) \geq \eta_{\psi(e)}(g_B)$.

Conversely, Suppose that $\tau_e(F^u(g_B)) \geq \eta_{\psi(e)}(g_B)$, for any $g_B \in \widetilde{(Y, K)}$. Consider $e_x^t \in dom(F)$ and $\eta_{\psi(e)}(g_B) \geq r$ such that $e_x^t \in F^u(g_B)$. Then $\tau_e(F^u(g_B)) \geq \eta_{\psi(e)}(g_B) \geq r$. Thus F is FS upper semi-continuous at e_x^t. So F is FS upper semi-continuous.

(2)The proof can be proved as (1).

(3) and (4) follow directly from (1), (2) and (X, τ_e) and $(Y, \eta_{\psi(e)})$ are two fuzzy topologies. □

The following example shows generally that F is FS upper semi-continuous but not normalized and $\tau_e(F^u(g_K)) \geq \eta_{\psi(e)}(g_B)$.

Example 4.4. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $\psi : E = \{e_1, e_2\} \to \{k_1, k_2\} = K$, where $\psi(e_1) = k_1$, $\psi(e_2) = k_2$. And let $F : X \multimap Y$ be a FS multifunction defined by:

$(G_F)_{k_1}(x_1, y_1) = 0.1$, $(G_F)_{k_1}(x_1, y_2) = 0.4$, $(G_F)_{k_1}(x_2, y_1) = 0.3$,
$(G_F)_{k_1}(x_2, y_2) = 0.2$, $(G_F)_{k_2}(x_1, y_1) = 0.4$, $(G_F)_{k_2}(x_1, y_2) = 0.6$,
$(G_F)_{k_2}(x_2, y_1) = 0.5$, $(G_F)_{k_2}(x_2, y_2) = 0.9$.

And let $f_E = \{(e_1, \{0.5, 0.8\}), (e_2, \underline{0.6})\}$, $g_K = \{(k_1, \{0.4, 0.6\}), (k_2, \{0.4, 0.5\})\}$ and the FS topologies (X, τ_E), (Y, η_K) are defined as:

$$\tau_e(u_E) = \begin{cases} 1 & \text{if} u_E = \Phi \text{ or } \tilde{\text{E}} \\ \frac{1}{2} & \text{if } u_E = f_E \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_k((w_K) = \begin{cases} 1 & \text{if } (w_K = \Phi \text{ or } \tilde{\text{K}} \\ \frac{1}{2} & \text{if } ; w_K = g_K \\ 0 & \text{otherwise.} \end{cases}$$

Then F is FS upper semi-continuous, where $F^u(g_K) = \{(k_1, \{0.6, 0.7\}), (k_2, \underline{0.5})\}$ and $dom(F) = \{(e_1, \{0.4, 0.3\}), (e_2, \{0.6, 0.9\})\}$. But $\tau_e(F^u(g_K))$=0$\ngeq \eta_{\psi(e)}(g_B)$=0.5 and F is not normalized.

The following example shows generally that F is fuzzy upper semi-continuous but not FS upper semi-continuous.

Example 4.5. Let $X=\{x_1,x_2\},\ Y=\{y_1,y_2\}$ and $\psi: E=\{e_1,e_2\}\to\{k_1,k_2\}=K$, where $\psi(e_1)=k_1,\ \psi(e_2)=k_2$. And let $F: X\multimap Y$ be a FS multifunction defined by:

$(G_F)_{k_1}(x_1,y_1)=0.6,\ (G_F)_{k_1}(x_1,y_2)=1.0,\ (G_F)_{k_1}(x_2,y_1)=1.0,$
$(G_F)_{k_1}(x_2,y_2)=0.1,\ (G_F)_{k_2}(x_1,y_1)=1.0,\ (G_F)_{k_2}(x_1,y_2)=0.5,$
$(G_F)_{k_2}(x_2,y_1)=0.4,\ (G_F)_{k_2}(x_2,y_2)=1.0$

Let $f_E=\{(e_1,\{0.4,0.3\}),(e_2,\{0.5,0.4\})\},\ g_E=\{(e_1,\{0.3,0.2\}),(e_2,\{0.4,0.3\})\},$ $h_K=\{(k_1,\{0.3,0.4\}),(k_2,\{0.4,0.3\})\}$ and FS topologies $(X,\tau_E),\ (Y,\eta_K)$ are defined as:

$$\tau_e(u_E)=\begin{cases}1 & \text{if} u_E=\Phi \ \text{ or } \ \tilde{\text{E}}\\ \frac{1}{2} & \text{if } u_E=f_E\\ \frac{1}{2} & \text{if } u_E=g_E\\ 0 & \text{otherwise,}\end{cases}$$

$$\eta_k((w_K)=\begin{cases}1 & \text{if } (w_K=\Phi \ \text{ or } \ \tilde{\text{K}}\\ \frac{1}{2} & \text{if } ;w_K=h_K\\ 0 & \text{otherwise.}\end{cases}$$

Then fuzzy topologies $(X,\tau_{e_1}),\ (X,\tau_{e_2}),\ (Y,\eta_{k_1}),\ (Y,\eta_{k_2})$ are defined as:

$$\tau_{e_1}(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0},\underline{1}\\ \frac{1}{2} & \text{if } \lambda=f_E(e_1)=\{0.4,0.3\}\\ \frac{1}{2} & \text{if } \lambda=g_E(e_1)=\{0.3,0.2\}\\ 0 & \text{otherwise,}\end{cases}$$

$$\tau_{e_2}(\lambda)=\begin{cases}1 & \text{if } \lambda=\underline{0},\underline{1}\\ \frac{1}{2} & \text{if } \lambda=f_E(e_2)=\{0.5,0.4\}\\ \frac{1}{2} & \text{if } \lambda=g_E(e_2)=\{0.4,0.3\}\\ 0 & \text{otherwise,}\end{cases}$$

$$\eta_{k_1}(\mu)=\begin{cases}1 & \text{if } \mu=\underline{0},\underline{1}\\ \frac{1}{2} & \text{if } \mu=h_K(k_1)=\{0.3,0.4\}\\ 0 & \text{otherwise,}\end{cases}$$

$$\eta_{k_2}(\mu)=\begin{cases}1 & \text{if } \mu=\underline{0},\underline{1}\\ \frac{1}{2} & \text{if } \mu=h_K(k_2)=\{0.4,0.3\}\\ 0 & \text{otherwise.}\end{cases}$$

Thus $dom(F)$=Ẽ and F is fuzzy upper semi-continuous, because

$$(F^u(h_K))(e_1)=\{0.4,0.3\}=f_E(e_1)$$

and

$$\tau_{e_1}((F^u(h_K))(e_1))=\frac{1}{2}\geq\eta_{\psi(e_1)}(h_K(\psi(e_1)))=\frac{1}{2}.$$

Also

$$(F^u(h_K))(e_2)=\{0.4,0.3\}=g_E(e_2)$$

and

$$\tau_{e_2}((F^u(h_K))(e_2))=\frac{1}{2}\geq\eta_{\psi(e_2)}(h_K(\psi(e_2)))=\frac{1}{2}.$$

But F is not FS upper semi-continuous, because

$$F^u(h_K) = \{(e_1, \{0.4, 0.3\}), (e_2, \{0.4, 0.3\})\}$$

and

$$\tau_e(F^u(h_K)) = 0 \not\geq \eta_{\psi(e)}(h_K) = \frac{1}{2}, \quad \forall\, e \in E.$$

Theorem 4.6. *Let $\varphi : X \to Y$, $\psi : E \to K$, $\vartheta : Y \to Z$ and $\varpi : K \to M$. Let $F : X \multimap Y$, $H : Y \multimap Z$ be two FS multifunctions and let (X, τ_E), (Y, η_K) and (Z, σ_M) be three FS topological spaces. Then we have the following:*

(1) *If F and H are normalized FS upper semi-continuous, then $H \circ F$ is FS upper semi-continuous,*

(2) *If F and H are FS lower semi-continuous, then $H \circ F$ is FS lower semi-continuous.*

Proof. (1) Let F, H be normalized FS upper semi-continuous and $w_D \in \widetilde{(Z, M)}$, $e \in E$. Then from Theorem 3.10, we have

$$\tau_e((H \circ F)^u(w_D)) = \tau_e((F^u(H^u(w_D)))) \geq \eta_{\psi(e)}(H^u(g_B)) \geq \sigma_{\varpi(\psi(e))}(w_D).$$

Thus $H \circ F$ is FS upper semi-continuous.

(2) The proof is similar to (1). □

Definition 4.7. Let $\psi : E \to K$ and $F : (X, \tau_E, \mathcal{I}) \multimap (Y, \eta_K)$ be a FS multifunction, where $\mathcal{I}$ is a FS ideal on X. For $\mathcal{A}, \mathcal{B}$ are FS operators on (X, τ_E) and $\mathcal{C}, \mathcal{D}$ are FS operators on (Y, η_K), respectively. Then, $\forall\, g_B \in \widetilde{(Y, K)}$, $r \in I_0$ and $e \in E$,

(i) F is called FS lower $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuous, if

$$\mathcal{I}_e[\mathcal{A}(e, F^l(\mathcal{D}(\psi(e), g_B, r)), r) \,\overline{\wedge}\, \mathcal{B}(e, F^l(\mathcal{C}(\psi(e), g_B, r)), r)] \geq \eta_{\psi(e)}(g_B).$$

(ii) suppose F is a normalized, then F is called FS upper $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{I})$-continuous, if

$$\mathcal{I}_e[\mathcal{A}(e, F^u(\mathcal{D}(\psi(e), g_B, r)), r) \,\overline{\wedge}\, \mathcal{B}(e, F^u(\mathcal{C}(\psi(e), g_B, r)), r)] \geq \eta_{\psi(e)}(g_B).$$

We can see that, the above definition is generalized of the concept of FS lower (resp. upper) semi-continuous multifunction (Theorem 4.3), when we choose, $\mathcal{A} =$ identity operator, $\mathcal{B} =$ interior operator, $\mathcal{D} =$identity operator, $\mathcal{C} =$identity operator and $\mathcal{I} = \mathcal{A}^0$.

From the above definition, we can present different cases of the FS continuity multifunction as follow:

(i) F is FS lower almost continuous (resp. normalized FS upper almost continuous) multifunction: for every $g_B \in \widetilde{(Y, K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \text{int}_\tau(e, F^l(\text{int}_\eta(\psi(e), \text{cl}_\eta(\psi(e), g_B, r), r)), r)$$

$$(\text{ resp.} \quad F^u(g_B) \sqsubseteq \text{int}_\tau(e, F^u(\text{int}_\eta(\psi(e), \text{cl}_\eta(\psi(e), g_B, r), r)), r))$$

Here, $\mathcal{A} =$identity operator, $\mathcal{B} =$ interior operator, $\mathcal{C} =$interior closure operator, $\mathcal{D} =$identity operator and $\mathcal{I} = \mathcal{I}^0$.

(ii) F is FS lower weakly continuous (resp. normalized FS upper weakly continuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \mathrm{int}_\tau(e, F^l(\mathrm{cl}_\eta(\psi(e), g_B, r)), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{int}_\tau(e, F^u(\mathrm{cl}_\eta(\psi(e), g_B, r)), r))$$

Here, $\mathcal{A}$=identity operator, $\mathcal{B}$=interior operator, $\mathcal{C}$= closure operator, $\mathcal{D}$=identity operator and $\mathcal{I} = \mathcal{I}^0$.

(iii) F is FS lower almost weakly continuous (resp. normalized FS upper almost weakly continuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^l(\mathrm{cl}_\eta(\psi(e), g_B, r)), r), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^u(\mathrm{cl}_\eta(\psi(e), g_B, r)), r), r))$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B}$ = interior closure operator, $\mathcal{C}$ = closure operator, $\mathcal{D}$ =identity operator and $\mathcal{I} = \mathcal{I}^0$.

(iv) F is FS lower precontinuous (resp. normalized FS upper precontinuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^l(g_B), r), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^u(g_B), r), r) \ )$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B}$ = interior closure operator, $\mathcal{C}$ = identity operator, $\mathcal{D}$ = identity operator and $\mathcal{I} = \mathcal{I}^0$.

(v) F is FS lower strongly precontinuous (resp. normalized FS upper strongly precontinuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{P\,cl}_\tau(e, F^l(g_B), r), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{P\,cl}_\tau(e, F^u(g_B), r), r) \ )$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B}$ = interior preclosure operator, $\mathcal{C}$ = identity operator, $\mathcal{D}$ = identity operator and $\mathcal{I} = \mathcal{I}^0$.

(vi) F is FS lower α-continuous (resp. normalized FS upper α-continuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then;

$$F^l(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, \mathrm{int}_\tau(e, F^l(g_B), r), r), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, \mathrm{int}_\tau(e, F^l(g_B), r), r), r))$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B}$ = interior closure interior operator, $\mathcal{C}$ = identity operator, $\mathcal{D}$ = identity operator and $\mathcal{I} = \mathcal{I}^0$.

(vii) F is FS lower almost α-continuous (resp. normalized FS upper almost α-continuous) multifunction: $\forall\ g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \alpha\,\mathrm{int}_\tau(e, F^l(\mathrm{S\,cl}_\eta(\psi(e), g_B, r)), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \alpha\,\mathrm{int}_\tau(e, F^u(\mathrm{S\,cl}_\eta(\psi(e), g_B, r)), r))$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B} = \alpha$ interior operator, $\mathcal{C}$ =semi-closure operator, $\mathcal{D} =$ identity operator and $\mathcal{I} = \mathcal{I}^0$.

(viii) F is FS lower weakly α-continuous (resp. normalized FS upper weakly α-continuous) multifunction: $\forall\ g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \alpha\,\mathrm{int}_\tau(e, F^l(\mathrm{cl}_\eta(\psi(e), g_B, r)), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \alpha\,\mathrm{int}_\tau(e, F^u(\mathrm{cl}_\eta(\psi(e), g_B, r)), r))$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B} = \alpha$ interior operator, $\mathcal{C}$ =closure operator, $\mathcal{D} =$ identity operator and $\mathcal{I} = \mathcal{I}^0$.

(ix) F is FS lower β-continuous (resp. normalized FS upper β-continuous) multifunction: for every $g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then;

$$F^l(g_B) \sqsubseteq \mathrm{cl}_\tau(e, \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^l(g_B), r), r), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \mathrm{cl}_\tau(e, \mathrm{int}_\tau(e, \mathrm{cl}_\tau(e, F^u(g_B), r), r), r))$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B} =$ closure interior closure operator, $\mathcal{C} =$ identity operator, $\mathcal{D} =$ identity operator and $\mathcal{I} = \mathcal{I}^0$.

(x) F is FS lower almost β-continuous (resp. normalized FS upper almost β-continuous) multifunction: $\forall\ g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \beta\,\mathrm{int}_\tau(e, F^l(\mathrm{S\,cl}_\eta(\psi(e), g_B, r)), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \beta\,\mathrm{int}_\tau(e, F^u(\mathrm{S\,cl}_\eta(\psi(e), g_B, r)), r)\ )$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B} = \beta$ interior operator, $\mathcal{C}$ =semi-closure operator, $\mathcal{D} =$ identity operator and $\mathcal{I} = \mathcal{I}^0$.

(xi) F is FS lower weakly β-continuous (resp. normalized FS upper weakly β-continuous) multifunction: $\forall\ g_B \in \widetilde{(Y,K)}$, $r \in I_0$ and $e \in E$ with $\eta_{\psi(e)}(g_B) \geq r$, then

$$F^l(g_B) \sqsubseteq \beta\,\mathrm{int}_\tau(e, F^l(\mathrm{cl}_\eta(\psi(e), g_B, r)), r)$$
$$(\text{ resp. } \quad F^u(g_B) \sqsubseteq \beta\,\mathrm{int}_\tau(e, F^u(\mathrm{cl}_\eta(\psi(e), g_B, r)), r)\ )$$

Here, $\mathcal{A}$ =identity operator, $\mathcal{B} = \beta$ interior operator, $\mathcal{C}$ =closure operator, $\mathcal{D} =$ identity operator and $\mathcal{I} = \mathcal{I}^0$.

Where, $\mathrm{int}_\tau, \mathrm{cl}_\tau, \mathrm{S\,cl}_\eta, \mathrm{P\,cl}_\eta, \alpha\,\mathrm{int}_\tau, \beta\,\mathrm{int}_\tau$ and $\beta\,\mathrm{cl}_\tau$ are found in ([1, 16]).

5. Conclusions

In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce the notions of fuzzy soft multifunction, upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we use these ideas to introduce upper and lower fuzzy soft semi-continuous multifunctions which are generalization of the concepts introduced in Abbas et al. [4, 5]. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

Acknowledgements. The authors would like to thank the referees for their valuable comments and suggestions which have improved this paper.

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